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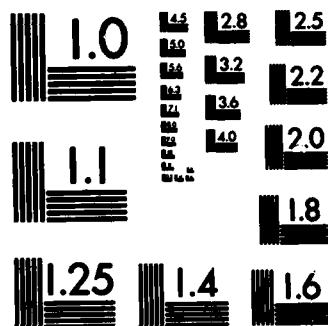
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## Numerical Solution of Inverse Problems

### Statement of Problems Studied

It was proposed to derive efficient 3 dimensional numerical algorithms for inversion of model equations: in geophysics, for underground detection; and in ultrasonic tomography, for accurate image reconstruction of a part of the interior of a human being.

An appropriate model for both of the above problems is the Helmholtz equation model

$$(1) \quad \nabla^2 u + \frac{\omega^2}{c^2(\mathbf{r})} u = 0 \quad ,$$

or equivalently

$$(2) \quad \nabla^2 u + k^2(1+f)u = 0 \quad ,$$

where

$$(3) \quad k = \frac{\omega}{c_0} \quad , \quad f(\mathbf{r}) = \frac{c_0^2}{c^2(\mathbf{r})} - 1 \quad .$$

This is the simplest model which contains all of the difficulties that one encounters, even in the inversion of more complicated models, such as the Maxwell equations.

The problem is to reconstruct the function  $f$  , based on measuring  $u$  on the boundary of the region.

### Summary of Most Important Results

We mention 3 results. The first two are important in applications, since they are the only economically feasible methods we know of, for inverting genuine 3-dimensional partial differential equations. Ours are not only economically feasible,

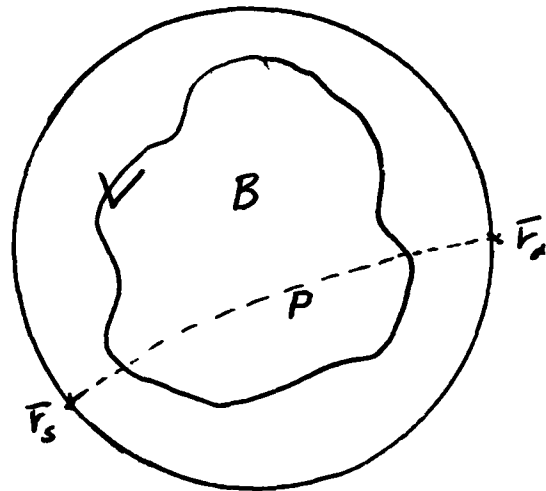
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they are also optimally efficient based on mathematical results [1] obtained under previous ARO sponsorship. The third result is important in the applications of mathematics, since it explains how, and when the well-known rational function algorithms work.

### §1.

In the case of ultrasonic tomography,  $u$  denotes the sound pressure,  $c(\bar{r})$  is the "speed" of sound in the body  $B$ ,  $c_0$  is the speed of sound in the medium (usually a liquid) surrounding  $B$ , and  $\omega$  denotes the frequency. In this case  $u = u_k$  satisfies the relation



$$(4) \quad \phi(k) \equiv \frac{1}{ik} \log u_k(\bar{r}) \Big|_{\bar{r}_s}^{\bar{r}_d} = \int_P \sqrt{1+f} \, ds + O(k^{-\sigma}), \quad k \rightarrow \infty$$

where  $P$  is the path of sound from the source point  $\bar{r}_s$  to the detector point  $\bar{r}_d$ . We have developed an effective method of reconstructing  $f$ , based on any model for which the sound pressure  $u_k$  satisfies (4). Our method predicts  $\phi(\infty) = \int_P \sqrt{1+f} \, ds$  via a simple rational function algorithm. In practice  $u_k(\bar{r}_s)$  and  $u_k(\bar{r}_d)$  may be measured for all frequencies (recall  $k = \omega/c_0$ ,  $\omega = 2\pi \times \text{frequency}$ ) in (e.g.) the range 1 megahertz  $\leq \omega/2\pi \leq 4$  megahertz. Our experiments have shown that if  $u_k(\bar{r}_s)$  and  $u_k(\bar{r}_d)$  are known accurately, then we can accurately reconstruct  $f$  via well-known x-ray algorithms. In practice one has noise, and there is error in the measurements. We have thus also developed a simple  $\ell^1$  procedure for circumventing the effects of these errors. We are thus able to distinguish objects having a diameter of about one wavelength and separated by about a wavelength.

A publication of these results will reach your office before Nov. 30, 1982.

§2.

The author has also developed a procedure for reconstructing the function  $f(\bar{r})$  in the equation (2) in the region  $\{\bar{r} = (x,y,z): z > 0\}$ , assuming that  $u_k(x,y,0)$  is known for  $0 \leq k \leq \infty$ . This latter procedure is based on a new formula for  $f$ , some new approximations, and sinc approximation formulas developed by the author [1] under previous ARO support. It has the desirable feature that it requires no matrix inversion. Rather, the 3-dimensional problem "decouples" and requires only the numerical approximation of one and two dimensional integrals. A paper on this procedure will reach your office before Dec. 20, 1982.

§3.

Probably the author's most important discovery from a mathematical standpoint is in the area of rational functions. We would not have been able to solve the problem §1 above without this result. We first describe a special case of this result, and then point out its consequences.

(a) If  $p > 1$ ,  $q = p/(p-1)$ , if  $g$  is analytic in the unit disc  $U$  of the complex plane, such that

$$(5) \quad \|G\|_p = \lim_{r \rightarrow 1^-} \left( \frac{1}{2\pi} \int_0^{2\pi} |G(re^{i\theta})|^p d\theta \right)^{1/p} < \infty$$

where  $G(z) = \frac{g(z)}{1-z^2}$ , then by taking

$$(6) \quad h = \pi \left( \frac{q}{2N} \right)^{1/2}, \quad S(k,h) \circ (x) = \frac{\sin\left[\frac{\pi}{h} (x-kh)\right]}{\frac{\pi}{h} (x-kh)}$$

$$(7) \quad z_k = (e^{kh} - 1)/(e^{kh} + 1)$$

$$(8) \quad \phi(x) = \log \left( \frac{1+x}{1-x} \right)$$

we have

$$(9) \quad |g(x) - \sum_{k=-N}^N g(z_k) S(k, h) \cdot \phi(x)| \leq C_p N^{1/2} \exp\{-\pi(\frac{N}{2q})^{1/2}\}.$$

The bound on the right hand side is essentially the best possible, under *any*  $2N + 1$  - point method of approximation.

This result [1] was discovered by the author under previous ARO sponsorship. While this result was reported earlier, we mention it here again, since it led to, and is related to the results (b), (c) which follow.

(b) Under the same conditions in (a) above one sets

$$(10) \quad \rho(z) = (1-z^2)^{\frac{N}{2}} \prod_{k=-N}^N \frac{z-z_k}{1-\bar{z}_k z}$$

then

$$(11) \quad |g(x) - \sum_{k=-N}^N \frac{\rho(x) g(z_k)}{(x-z_k)\rho'(z_k)}| \leq C'_p N^{1/2} \exp\{-\pi(\frac{N}{2q})^{1/2}\}.$$

That is, we have an explicit rational approximation of  $g$  for which the error converges at the same optimal rate as that in (9). Because of its optimal nature this linear rational approximation procedure is very nearly as good as the best rational approximation of  $g$ , of the same degree. It is curious that the (almost) optimal interpolation points are the "sinc" points (7).

(c) Other consequences of the result (b):

- (i) It is easy to develop a Thiele algorithm which yields the rational function in (11).
- (ii) The problem of poles on the interval of approximation, which has plagued users of rational function in the past, has been eliminated.
- (iii) We have described a space of functions in which rational approximation works. We can thus tell a priori, when e.g. the "Rho-algorithm" for prediction, or the ODE method of Bulirsch and Stoer work.



(iv) The transformation

$$(12) \quad z = \frac{\xi - 1}{\xi + 1}$$

is a conformal map of the unit disc onto the right half plane. It also transforms the rational function in (11) into another rational function for interpolation over  $(0, \infty)$ , and a description of a space of functions (e.g., analyticity and boundedness on the right-half plane, of class  $\text{Lip}_{(1/q)}$  on  $(0, \infty)$ ) for which this works. The corresponding rational function approximation is

$$(13) \quad G(\xi) \cong \sum_{k=-N}^N \frac{\psi(\xi) G(\xi_k)}{(\xi - \xi_k) \psi'(\xi_k)}, \quad \psi(\xi) = \frac{\xi}{1+\xi} \prod_{k=-N}^N \frac{\xi - \xi_k}{\xi + \xi_k}$$

where  $\xi_k = e^{kh}$ , and the error is  $O[N^{1/2} \exp\{-\pi(\frac{N}{2q})^{1/2}\}]$ , and where we have also assumed that  $G(0) = G(\infty) = 0$ .

(v) Similary  $\xi = e^w$  yields a rational function of  $e^w$ ; it describes a space of function for which this is an optimal approximation, and in particular, it describes a space, and a set of interpolation points ( $w_k = kh$ ) for which P. Wynn's "epsilon algorithm" - which includes the Pade method - always works.

We hope to have these results in your office by the end of January, 1983.

(vi) Our result plays the same role for rational approximation as the Chebyshev polynomials play for approximation by polynomials. If  $F$  is analytic in the ellipse  $E_R$  with foci at  $\pm 1$  and sum of semi-axes equal to  $R$  ( $R > 1$ ) then Lagrange interpolation at the nodes of the Chebyshev polynomial  $T_{n+1}$  produces very nearly as good a polynomial of degree  $n$  approximation to  $F$  on  $[-1, 1]$  as the best (uniform) polynomial approximation. Since the best

uniform polynomial approximation is unduly difficult to obtain, one is almost always satisfied with the nearly best approximation provided by the Chebyshev polynomial.

#### References

- [1] Stenger, F., *Numerical methods based on the Whittaker Cardinal, or Sinc Functions*, SIAM Review 23(1981), 165-224.

#### List of Publications

Two papers published:

- (a) *Asymptotic Ultrasonic Inversion Based on Using More than One Frequency*, in *Acoustical Imaging II* (1982) Plenum Press, pp. 425-444. (Submitted to your office previously.)
- (b) *Wave Equations and Inverse Solutions for Soft Tissue* (with S. A. Johnson, C. Wilcox, J. Ball, M. J. Berggren) in *Acoustical Imaging II* (1982) Plenum Press, pp. 409-424. (Submitted to your office previously.)

#### Advanced Degrees Earned

K. Sikorski, Ph.D., University of Utah (1982).

